TUT 3: EXACT EQUATIONS AND INTEGRATING FACTORS

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Up to now, we have reviewed the methods to solve first order linear ODEs by using integrating factor, to solve separable equations and use transformation method to solve homogeneous equations and Bernoulli equations. In this tutorial, I am going to recap two more general equations, called exact equations and the one with integrating factors. In the end, I give a summary of all the methods you have learned to solve first order ODEs.

1. Exact equations

We know the general first order ODE is in the form

$$\frac{dy}{dx} = F(x, y),$$

while sometimes it can be written as

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad \text{or} \quad M(x, y)dx + N(x, y)dy = 0. \tag{1}$$

Definition: A equation with the above form called an **Exact Equation** if there exists a (potential) function $\Psi(x, y)$ such that

$$\partial_x \Psi(x, y) = M(x, y); \quad \partial_y \Psi(x, y) = N(x, y). \tag{2}$$

Going back to (1) and taking y as a function of x, then chain rule implies

$$\frac{d}{dx}\Psi(x, y(x)) = \partial_x \Psi(x, y) + \partial_y \Psi(x, y)\frac{dy}{dx} = M + N\frac{dy}{dx} = 0,$$

which gives $\Psi(x, y) \equiv c$ for some constant c. This gives the solution of the ODE!

But for a general equation (1), **how to determine an exact equation?** If you remember Green's theorem, you should know: for $(x, y) \in R = (\alpha, \beta) \times (\gamma, \delta)$, if $M(x, y), N(x, y)$ are continuous functions with continuous partial derivatives, the following statements are equivalent.

(a). There exists a function Ψ such that

$$d\Psi = \Psi_x dx + \Psi_y dy = Mdx + Ndy.$$

(b). For all $(x, y) \in R$,

$$M_y(x, y) = N_x(x, y).$$

Note that if (a) holds, the second condition is equivalent to $\Psi_{xy} = M_y = N_x = \Psi_{yx}$.

Remark 1. *Separable equations are also exact equations with*

$$M = M(x), \quad N = N(y) \quad \textit{and hence} \quad M_y = N_x = 0.$$

Now you know how to determine an exact equation, but to solve the equation, you need to find the potential function Ψ. In the following, let's see how should we proceed by an example.

Example 1. *Find the solution of*

$$9x^2 + y - 1 - (4y - x)y' = 0 \tag{3}$$

Solution.

Step 1. (Check $M_y = N_x$) With

$$M = 9x^2 + y - 1, \qquad N = -(4y - 1),$$

we find $M_y = N_x = 1$, so the given equation is exact. Therefore there is a function Ψ such that

$$\Psi_x = 9x^2 + y - 1, \qquad \Psi_y = -(4y - x). \tag{4}$$

Step 2. (Find the potential function Ψ) Firstly, integrating the first equation with respect to variable x gives

$$\Psi = 3x^3 + (y - 1)x + h(y).$$

Taking derivative with respect to y, then the second equation in (4) implies

$$\Psi_y = x + h'(y) = -(4y - x), \quad \text{and then} \quad h'(y) = -4y$$

Integrating gives

$$h(y) = -2y^2, \quad \text{and hence} \quad \Psi(x, y) = 3x^3 + (y - 1)x - 2y^2.$$

Step 3. (Determine the solution) In conclusion, the solution is

$$\Psi(x, y) = 3x^3 + (y - 1)x - 2y^2 = c$$

for some constant c.

2. Exact equations with Integrating factor

In general, the equation may not exactly in exact form.

Example 2. *Consider*

$$y(1 + x^2)dx + y^2dy = 0. \tag{5}$$

With

$$M = y(1 + x^2); \quad N = y^2,$$

We check

$$M_y = 1 + x^2 \neq 0 = N_x,$$

which means the equation is not exact. But the separable equation

$$(1 + x^2)dx + y^2dy = 0 \tag{6}$$

is exact, which can be solved by finding the potential function Ψ. So we can solve the equation (5) by solving

$$y = 0 \quad \textit{or} \quad (1 + x^2)dx + y^2dy = 0. \tag{7}$$

Conversely, it is reasonable to think if we multiply a suitable integrating factor $\mu(x, y)$ to the general equation, then it may become exact. We hope

$$\mu M + \mu N \frac{dy}{dx} = 0 \quad \text{or} \quad \mu M dx + \mu N dy = 0. \tag{8}$$

is an exact solution. Denote

$$\tilde{M} = \mu M, \qquad \tilde{N} = \mu N.$$

There holds that

$$\partial_y \mu \, M + \mu M_y = \partial_y \tilde{M} = \partial_x \tilde{N} = \partial_y \mu \, N + \mu N_y,$$

which can be rewritten as

$$M \partial_y \mu - N \partial_x \mu = \mu (N_y - M_x).$$

This is an linear PDE, you can not solve it in general. But it is possible to give a solution in some special situations.

(1). If $\mu = \mu(x)$ holds, then $\partial_y \mu = 0$ and

$$\partial_x \ln |\mu| = \frac{\partial_x \mu}{\mu} = \frac{M_x - N_y}{N},$$

the right hand side should independent of y. Conversely, if the right hand side depend only on x, then we can find a solution of the form $\mu = \mu(x)$.

(2). If $\mu = \mu(y)$ holds, then $\partial_x \mu = 0$ and

$$\partial_x \ln |\mu| = \frac{\partial_x \mu}{\mu} = \frac{N_y - M_x}{M},$$

the right hand side should independent of x. Conversely, if the right hand side depend only on y, then we can find a solution of the form $\mu = \mu(y)$.

(3). If $\mu = \mu(z) = \mu(xy)$ holds, then $\partial_x \mu = y\mu'$, $\partial_y \mu = x\mu'$ (here $\mu' = \dfrac{d\mu}{dz}$) and

$$\frac{d \ln |\mu|}{dz} = \frac{N_y - M_x}{xM - yN},$$

The right hand side should be a function of $z = xy$. Conversely, if the right hand side depend only on z, then we can find a solution of the form $\mu = \mu(z)$.

Remark 2. *First order linear ODEs can be solved by introducing an integrating factor $\mu = \mu(x)$.*

Example 3. *Find the solution of*

$$e^x + (e^x \cot y + 2y \csc y) y' = 0 \tag{9}$$

Solution. We only need to consider where $y \neq k\pi (k \in \mathbb{Z})$.

Step 1. (Check the equation is exact or not) With

$$M = e^x, \qquad N = e^x \cot y + 2y \csc y.$$

Since $M_y = 0 \neq e^x = N_x$, the equation is not exact.

Step 2. (Find the integrating factor to construct an exact equation) We introduce an integrating factor $\mu(x, y)$ such that

$$\tilde{M}dx + \tilde{N}dy := \mu M dx + \mu N dy = 0 \tag{10}$$

is an exact equation. Then

$$\tilde{M}_y = \partial_y \mu \, e^x = \partial_x \mu (e^x \cot y + 2y \csc y) + \mu e^x \cot y = \tilde{N}_x. \tag{11}$$

Let $\mu = \mu(y)$, then

$$\partial_y \mu = \mu \cot y$$

Hence we can choose

$$\mu = \sin y.$$

Hence with this μ, the equation (10) is an exact equation. Therefore there is a function $\Psi(x, y)$ such that

$$\Psi_x = \tilde{M}; \qquad \Psi_y = \tilde{N}.$$

Step 3. (Solve Ψ) Integrating the first equation with respect to x implies

$$\Psi(x, y) = \int \sin y \, e^x dx = e^x \sin y + h(y).$$

Taking derivative with respect to y

$$\Psi_y = e^x \cos y + h'(y) = \tilde{N} = \sin y (e^x \cot y + 2y \csc y).$$

Hence

$$h'(y) = 2y, \quad \text{and then} \quad h(y) = y^2,$$

which gives

$$\Psi(x, y) = e^x \sin y + y^2.$$

Step 4. (Find the solution) So the solution for (10) is

$$e^x \sin y + y^2 = c \tag{12}$$

for some constant c.

Remark 3. *(1) In this case we have $\mu(y) = \sin y = 0$ for $y = k\pi$, which is okay since the equation is not meaningful at $y = k\pi$.*

(2) In Example 2, we actually have $\mu = 1/y$, which is not meaningful at $y = 0$. And $y = 0$ is a solution to (5) *while it is not a solution to* (6)*. This means solving the equation after multiplying an integrating factor does not always solve the original equation, you need to check the singular curves.*

Exercise 4. *Find the solution of*

$$3x + \frac{6}{y} + \left(\frac{x^2}{y} + \frac{3y}{x}\right) y' = 0 \tag{13}$$

Solution. We only need to consider where $xy \neq 0$. Denote

$$M = 3x + \frac{6}{y}, \qquad N = \frac{x^2}{y} + \frac{3y}{x}.$$

By calculations,

$$M_y = -\frac{6}{y^2} \neq \frac{2x}{y} - \frac{3y}{x^2} = N_x.$$

Hence this equation is not an exact equation. Let $\mu = \mu(x, y)$ such that

$$\mu M + \mu N y' = 0$$

is an exact equation. Then

$$\partial_y \mu M + \mu M_y = \partial_y(\mu M) = \partial_x(\mu N) = \partial_x \mu N + \mu N_x,$$

which is

$$\partial_y \mu \left(3x + \frac{6}{y}\right) - \frac{6}{y^2}\mu = \partial_x \mu \left(\frac{x^2}{y} + \frac{3y}{x}\right) + \mu \left(\frac{2x}{y} - \frac{3y}{x^2}\right). \tag{14}$$

Since $\mu = \mu(x)$ or $\mu(y)$ are not solutions, we consider $\mu = \mu(z) = \mu(xy)$. Then

$$\partial_x \mu = y\mu', \qquad \partial_y \mu = x\mu'.$$

(14) can be reduced to

$$(\ln|\mu|)' = \frac{\mu'}{\mu} = \frac{1}{xy} = \frac{1}{z}. \tag{15}$$

Therefore, we can take

$$\mu = z = xy.$$

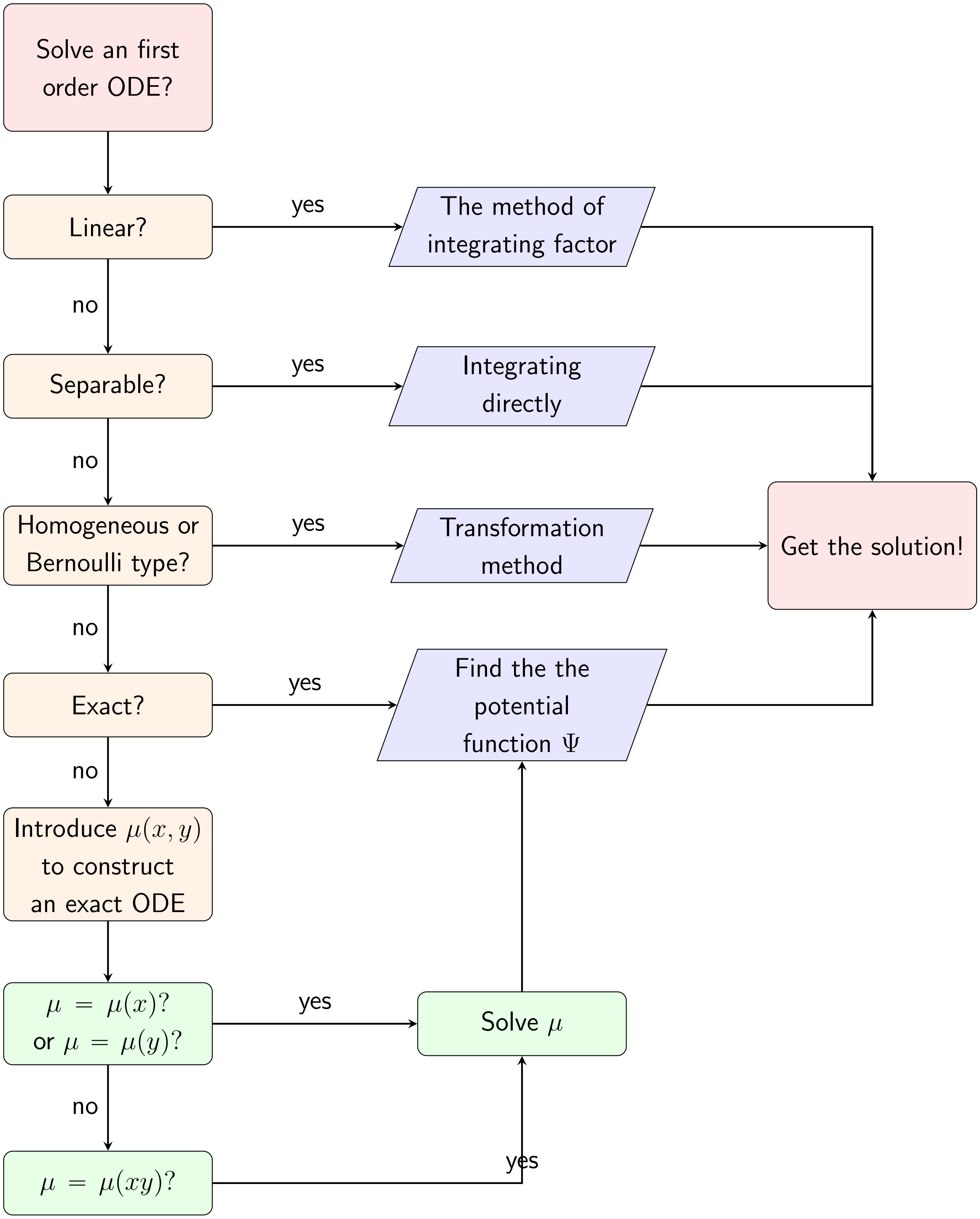
Now we get the new equation

$$3x^2y + 6x + (x^3 + 3y^2)y' = 0,$$

which is exact. You can solve it by finding the potential function Ψ.

3. Summary

Now let me give a summary about how to solve an first oder ODE. When you are going to solve a first order ODE, always ask yourself the following three questions first: is it linear? is it separable? is it homogeneous or Bernoulli type? If it is linear, then use the method of integrating factor; if it is separable, you can solve it directly; if it is homogeneous or Bernoulli, then first transform it into a separable equation. If none of the above is true, then you check if it is exact. Moreover, if it is not exact, then you should try to introduce the integrating factor. The flow chart is as following.



Exercise 5. *Take a look at the equations below, and determine what kind of method you will use to solve them following the flow chart above.*

(1). $y'e^x = (1+y)\sin x + \cot x;$
(2). $y' = xy + y^2;$
(3). $y' = ky + ry^2$ *for some constants* $k, r \neq 0;$
(4). $y'(\ln|y| - \ln|x|)x = y + x;$
(5). $y'\tan x = (\csc x + \sin x)\, e^{y+x};$
(6). $(3x - e^x \sin y)y' + (3y + e^x \cos y) = 0;$
(7). $y + (2x - ye^y)y' = 0;$
(8). $(x + 1/y)\, y' + (y + 1/x) = 0$

Solution.

(1). Rewrite the equation as

$$y' = ye^{-x}\sin x + e^{-x}(\sin x + \cot x),$$

which is a linear ODE. We can solve it by introducing an integrating factor.

(2). Firstly, this is not a linear equation since we have a term y^2; Secondly, it is not a separable equation, since $(xy + y^2)$ is not separable; It is actually an Bernoulli type equation, we can solve it by transformation method (introduce $z = y^{-1}$ and reduce to an linear equation).

(3). Clearly, it is not linear; It is an separable equation since the right hand side just depend on y. You can rewrite it as

$$\frac{1}{ky + ry^2}y' = 1 \quad \text{for some constants } k, r \neq 0.$$

Integrate directly, you can solve the equation. (This is also an Bernoulli equation.)

(4). Obviously, it is nonlinear and non-separable, also not Bernoulli type; Now we check if it is an homogeneous equation. Rewrite the equation as

$$y' \ln\left|\frac{y}{x}\right| = \frac{y + x}{x} = \frac{y}{x} + 1 \quad \text{or} \quad y' = \frac{y/x + 1}{\ln|y/x| + 1},$$

which is an homogeneous equation. You can introduce $z = y/x$, and reduce it to a separable equation.

(5). Obviously, it is nonlinear; Rewrite the equation as

$$y' = \frac{\csc x + \sin x}{\tan x} e^x e^y \quad \text{or} \quad e^{-y}dy = \frac{\csc x + \sin x}{\tan x} e^x dx,$$

which is separable.

(6). Note that this equation is nonlinear, non-separable, also not Bernoulli type or homogeneous equation. Hence we check whether it is an exact equation. With

$$M = 3y + e^x \cos y; \quad N = 3x - e^x \sin y.$$

We have

$$M_y = 3 - e^x \sin y = N_x.$$

Therefore, it is an exact equation, and you can solve it by finding the potential function Ψ.

(7). Note that this equation is nonlinear, non-separable, also not Bernoulli type or homogeneous equation. Hence we check whether it is an exact equation. With

$$M = y; \quad N = 2x + \frac{1}{y},$$

we have

$$M_y = 1 \neq 2 = N_x.$$

So the equation is not exact. We introduce an integrating factor μ such that

$$\tilde{M}dx + \tilde{N}dy =: \mu M dx + \mu N dy = 0$$

is an exact equation. Hence it holds that

$$\partial_y \mu M + \mu M_y = \tilde{M}_y = \tilde{N}_x = \partial_x \mu N + \mu N_x.$$

Or

$$\partial_y \mu \, y + \mu = \partial_x \mu \left(2x + \frac{1}{y}\right) + 2\mu.$$

We can check whether $\mu = \mu(x)$ or $\mu(y)$ is a special solution. Firstly, if $\mu = \mu(x)$, then

$$\mu = \partial_x \mu \left(2x + \frac{1}{y}\right) + 2\mu,$$

which is impossible since the right hand side depend on y. If $\mu = \mu(y)$, then

$$y\partial_y \mu = \mu \quad \text{and} \quad \mu = y.$$

Hence we know $y^2 + (2x - ye^y)yy' = 0$ is an exact equation, and you can solve it by finding the potential function Ψ.

(8). This is an similar problem to exercise 4.